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István Ozsváth

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Dust-Filled Universes of Class II and Class III*

István Ozsváth

Division of Mathematics, The University of Texas at Dallas, Dallas, Texas 75320

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I construct on the Lie group $R \times H^3$ two different families of left-invariant metrics which satisfy the Einstein field equations with incoherent matter, calling the Riemannian spaces M_4, obtained this way, Class II and Class III universes. We discuss the geometry of these universes.

1. INTRODUCTION

Schücking and I discussed recently[1] the physical and the geometrical properties of the finite rotating universe, the Class I solution, according to the terminology introduced by Farnsworth and Kerr.[2] The Class I solution is a family of left-invariant metrics on the Lie group $R \times S^3$ satisfying Einstein's field equations with dust.

In this paper, I discuss in a similar manner the Class II and Class III universes, which are two different families of metrics imposed on the same manifold, namely, one the Lie group $R \times H^3$. The Class IV universes given in Ref. 3 receive their treatment in a subsequent paper. The four classes exhaust all the possibilities of homogeneous dust solutions of Einstein's field equations as Refs. 2 and 3 show.

In order to keep this paper readable independently of Ref. 1, I repeat some general remarks made there and suggest that the reader glance at Ref. 1 too.

2. USEFUL THEOREMS AND FORMULAS

As a technical introduction we list some well-known theorems and formulas for later use.[4]

Given a 4-dimensional manifold M_4, we denote the vector fields on M_4 by $X, Y, Z, \cdots$ and the 1-forms by $\omega, \theta, \phi, \cdots$. M_4 and the tensor fields can be regarded as analytic. The exterior derivative of ω is given by

$$d\omega(X, Y) = \tfrac{1}{2}\{X\omega(Y) - Y\omega(X) - \omega([X, Y])\}. \quad (2.1)$$

We denote the basis for the vector fields by

$$X_0, X_1, X_2, X_3 \quad (2.2)$$

and that for the 1-forms by

$$\omega^0, \omega^1, \omega^2, \omega^3, \quad (2.3)$$

where

$$\omega^a(X_b) = \delta^a{}_b. \quad (2.4)$$

We introduce an affine connection on M_4 by

$$\nabla_{X_a}(X_b) = \Gamma_{ab}{}^c X_c. \quad (2.5)$$

The connection form is defined by

$$\omega^a{}_b = \Gamma_{cb}{}^a\, \omega^c. \quad (2.6)$$

The covariant differentiation ∇_X is a derivation of the algebra $T(M_4)$ of the tensor fields such that it preserves the type of the tensor field and commutes with all contractions. The covariant derivative of a vector field Y is given by

$$\nabla_X(Y) = \xi^a(X_a\eta^c + \eta^b\Gamma_{ab}{}^c)X_c, \quad (2.7)$$

where

$$X = \xi^a X_a \quad \text{and} \quad Y = \eta^b X_b. \quad (2.8)$$

The covarient derivative of a 1-form is

$$U(X, Y) = (\nabla_X\omega)(Y) = X\omega(Y) - \omega(\nabla_X(Y)). \quad (2.9)$$

The components of the tensor field U, defined above, are given by

$$U_{ab} = U(X_a, X_b) = X_a u_b - \Gamma_{ab}{}^c u_c, \quad (2.10)$$

where

$$u_b = \omega(X_b). \quad (2.11)$$

The Lie derivative of Y with respect to X is defined by

$$L_X(Y) = \nabla_X(Y) - \nabla_Y(X). \quad (2.12)$$

One defines the torsion tensor field by

$$T(X, Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y] \quad (2.13)$$

and the curvature tensor field by

$$R(X, Y)Z = \nabla_X\nabla_Y(Z) - \nabla_Y\nabla_X(Z) - \nabla_{[X,Y]}(Z). \quad (2.14)$$

The components of T and R are

$$T(X_b, X_c) = T^a{}_{bc} X_a, \quad R(X_c, X_d)X_b = R^a{}_{bcd} X_a. \quad (2.15)$$

Cartan's structure equations are

$$d\omega^a = -\omega^a{}_p \wedge \omega^p + \tfrac{1}{2}T^a{}_{pq}\, \omega^p \wedge \omega^q, \quad (2.16)$$

$$d\omega^a{}_b = -\omega^a{}_p \wedge \omega^p{}_b + \tfrac{1}{2}R^a{}_{bpq}\, \omega^p \wedge \omega^q. \quad (2.17)$$

We assume henceforth that

$$T = 0, \quad \text{that is,} \quad T^a{}_{bc} = 0. \tag{2.18}$$

We define the functions

$$C^a{}_{bc} = -C^a{}_{cb}, \quad a, b, c, \cdots = 0, 1, 2, 3, \tag{2.19}$$

by the equations

$$[X_b, X_c] = C^a{}_{bc} X_a; \tag{2.20}$$

then it follows, by using (2.1) and (2.4), that

$$d\omega^a = -\tfrac{1}{2} C^a{}_{pq}\, \omega^p \wedge \omega^q \tag{2.21}$$

and, from (2.16), (2.18) and (2.6), that

$$d\omega^a = -\tfrac{1}{2}(\Gamma_{pq}{}^a - \Gamma_{qp}{}^a)\omega^p \wedge \omega^q. \tag{2.22}$$

Therefore

$$C^a{}_{bc} = \Gamma_{bc}{}^a - \Gamma_{cb}{}^a. \tag{2.23}$$

We introduce a pseudo-Riemannian metric on M_4 by the nondegenerate tensor field

$$g(X, Y) = g(Y, X). \tag{2.24}$$

It is well known that on a pseudo-Riemannian manifold there exists one and only one affine connection such that

$$T = 0 \quad \text{and} \quad \nabla_Z g = 0, \tag{2.25}$$

that is,

$$\nabla_X(Y) - \nabla_Y(X) = [X, Y] \tag{2.26}$$

and

$$\begin{aligned} 2g(\nabla_X(Y).Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & + g(Y, [Z, X]) + g(Z, [X, Y]) \\ & - g(X, [Y, Z]). \end{aligned} \tag{2.27}$$

Suppose that

$$g(X_a, X_b) = g_{ab} = \operatorname{diag}(+1, -1, -1, -1), \tag{2.28}$$

in other words,

$$g = g_{ab}\omega^a\omega^b. \tag{2.29}$$

It follows then that

$$\begin{aligned} 2g(\nabla_{X_a}(X_b), X_c) = {} & g(X_b, [X_c, X_a]) + g(X_c, [X_a, X_b]) \\ & - g(X_a, [X_b, X_c]) \end{aligned} \tag{2.30}$$

and, using the notation

$$\Gamma_{abc} = \Gamma_{ab}{}^d g_{dc}, \quad C_{abc} = g_{ad} C^d{}_{bc}, \tag{2.31}$$

we obtain

$$\Gamma_{abc} = \tfrac{1}{2}(C_{bca} + C_{cab} - C_{abc}). \tag{2.32}$$

Using (2.1) and (2.17), we have

$$\begin{aligned} & d\omega^a{}_b(X_c, X_d) \\ & = \tfrac{1}{2}(X_c\omega^a{}_b(X_d) - X_d\omega^a{}_b(X_c) - \omega^a{}_b([X_c, X_d])) \\ & = -\tfrac{1}{2}(\omega^a{}_p(X_c)\omega^p{}_b(X_d) - \omega^a{}_p(X_d)\omega^p{}_b(X_c)) + \tfrac{1}{2}R^a{}_{bcd}, \end{aligned}$$

and, therefore,

$$\begin{aligned} R^a{}_{bcd} = {} & \Gamma_{cf}{}^a\, \Gamma_{db}{}^f - \Gamma_{df}{}^a\, \Gamma_{cb}{}^f - \Gamma_{fb}{}^a\, C^f{}_{cd} \\ & + X_c\Gamma_{db}{}^a - X_d\Gamma_{cb}{}^a. \end{aligned} \tag{2.33}$$

It should be noted that the power of the formalism developed above lies in the freedom of choice for the basis X_0, X_1, X_2, X_3 of the vector fields or $\omega^0, \omega^1, \omega^2, \omega^3$ of the 1-forms, respectively [with the proviso (2.4)]. In the following we specialize our manifold M_4 and make a definite choice for the case most adequate for our problem. The steps are as follows: Suppose that the functions $C^a{}_{bc}$ are constants and satisfy the Jacobi identities. Then our pseudo-Riemannian manifold M_4 is a Lie group. Suppose that M_4 is simply connected. Then it is the universal covering group, uniquely defined by the Lie algebra (2.20) of the invariant vector fields X_0, X_1, X_2, X_3. The corresponding left-invariant 1-forms $\omega^0, \omega^1, \omega^2, \omega^3$ satisfy

$$d\omega^a = -\tfrac{1}{2} C^a{}_{bc}\omega^b \wedge \omega^c \tag{2.34}$$

and characterize M_4 equivalently.

We choose now, for the base of vector fields or of the 1-forms on M_4, the invariant vector fields X_0, X_1, X_2, X_3 or invariant 1-forms $\omega^0, \omega^1, \omega^2, \omega^3$, respectively.

The requirement (2.28), that the X_0, X_1, X_2, X_3 should be pseudo-orthonormal, defines the left-invariant metric. Or, equivalently,

$$g = g_{ab}\omega^a\omega^b. \tag{2.35}$$

Generally speaking, this choice of base and the formalism sketched above allows one to discuss many properties of the group or Riemannian space M_4 from a simple knowledge of the constants of structure $C^a{}_{bc}$. We do not have to specialize the coordinates and can perform many calculations without an explicit knowledge of the left-invariant forms. The most important formal consequence of the above choice is that the corresponding Γ's are constants [see (2.32)]. But, above all in importance, our results will be global results since the theory of Lie groups[5] assures us that these vector fields and forms exist globally. Since the Γ's are constants, (2.33) reduces to

$$R^a{}_{bcd} = \Gamma_{cf}{}^a\, \Gamma_{db}{}^f - \Gamma_{df}{}^a\, \Gamma_{cb}{}^f - \Gamma_{fb}{}^a\, C^f{}_{cd}. \tag{2.36}$$

The components of the Ricci tensor field are

$$R_{bc} = R^f{}_{bcf} = \Gamma_{fb}{}^g\, \Gamma_{gc}{}^f + C^g{}_{fg}\, \Gamma_{bc}{}^f. \tag{2.37}$$

The field equations are

$$\begin{aligned} G_{ab} + \Lambda g_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = -\kappa\rho u_a u_b, & \\ u_a u^a = 1, & \end{aligned} \tag{2.38}$$

where

$$R = R^a{}_a. \tag{2.39}$$

After a trivial computation, we obtain

$$R_{ab} = -\kappa\rho u_a u_b + (\Lambda + \kappa\rho/2)g_{ab}, \quad u_a u^a = 1. \tag{2.40}$$

3. THE GROUP

Consider a 4-dimensional vector space over the field of real numbers. We denote the base vectors by

$$\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \tag{3.1}$$

and convert this vector space into an algebra by introducing the noncommutative multiplication by the following requirements:

$$\begin{aligned} &\mathbf{e}_0\mathbf{e}_\mu = \mathbf{e}_\mu\mathbf{e}_0 = \mathbf{e}_\mu, \quad \mu = 0, 1, 2, 3, \\ &\mathbf{e}_1\mathbf{e}_1 = -\mathbf{e}_0, \quad \mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_0, \quad \mathbf{e}_3\mathbf{e}_3 = \mathbf{e}_0, \\ &\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_2 = -\mathbf{e}_1, \quad \mathbf{e}_3\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2, \\ &\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3. \end{aligned} \tag{3.2}$$

We call this algebra Gödel's quaternion algebra and the vectors

$$\mathbf{a} = a^\mu \mathbf{e}_\mu \tag{3.3}$$

Gödel quaternions.[6]

Introducing the conjugate quaternion $\mathbf{a}^*$ by

$$\mathbf{a}^* = a^0\mathbf{e}_0 - a^i\mathbf{e}_i, \tag{3.4}$$

we have from (3.2) that

$$\begin{aligned} \mathbf{a}\mathbf{a}^* &= [(a^0)^2 + (a^1)^2 - (a^2)^2 - (a^3)^2]\mathbf{e}_0 \\ &= (a^0)^2 + (a^1)^2 - (a^2)^2 - (a^3)^2. \end{aligned} \tag{3.5}$$

We identified here the subfield $a^0\mathbf{e}_0$ with the real field.

Consider now the normed Gödel quaternions, that is, quaternions $\mathbf{a}$ satisfying the condition

$$\mathbf{a}\mathbf{a}^* = (a^0) + (a^1)^2 - (a^2)^2 - (a^3)^2 = 1. \tag{3.6}$$

They obviously form a group with respect to the quaternion multiplication. Identifying the vectors (3.1) with the unit vectors along the axes in a 4-dimensional pseudo-Euclidean space of signature

$$+ + - -, \tag{3.7}$$

or Euclidean space of coordinates a^0, a^1, a^2, a^3, denoted by E^4, we find that (3.6) is the equation of the sphere or hyperboloid H^3, respectively. The manifold H^3 with the quaternion multiplication (3.2) is a Lie group, which we denote also by H^3.

We want to obtain the left-invariant vector fields of H^3 in the coordinate system induced by the Cartesian coordinates of the imbedding E^4. We consider, therefore, in the point $\mathbf{e}_0$ on H^3 the three vectors

$$\mathbf{e}_0 + \epsilon\mathbf{e}_i, \quad i = 1, 2, 3, \tag{3.8}$$

tangential to H^3 and propagate them by the left translations over H^3 generating the three independent left-invariant vector fields mentioned above. Since

$$\mathbf{a} = \mathbf{a}\mathbf{e}_0, \quad \mathbf{a} + \epsilon\boldsymbol{\omega}_i = \mathbf{a}(\mathbf{e}_0 + \epsilon\mathbf{e}_i), \tag{3.9}$$

we obtain

$$\boldsymbol{\omega}_i = \mathbf{a}\mathbf{e}_i \tag{3.10}$$

as the vectors at $\mathbf{a}$, corresponding to $\mathbf{e}_i$ at $\mathbf{e}_0$. Defining the components $e_i{}^\mu$ of $\boldsymbol{\omega}_i$ by

$$\boldsymbol{\omega}_i = e_i{}^\mu \mathbf{e}_\mu, \tag{3.11}$$

we obtain, using (3.10), (2.3), and (3.3), the following expressions:

$$\begin{aligned} &e_1{}^\mu = (-a^1, a^0, a^3, -a^2), \quad e_2{}^\mu = (a^2, a^3, a^0, a^1), \\ &e_3{}^\mu = (a^3, -a^2, -a^1, a^0). \end{aligned} \tag{3.12}$$

Therefore, the invariant vector fields

$$E_i = e_i{}^\mu \frac{\partial}{\partial a^\mu} \tag{3.13}$$

are given by

$$\begin{aligned} E_1 &= -a^1\frac{\partial}{\partial a^0} + a^0\frac{\partial}{\partial a^1} + a^3\frac{\partial}{\partial a^2} - a^2\frac{\partial}{\partial a^3}, \\ E_2 &= a^2\frac{\partial}{\partial a^0} + a^3\frac{\partial}{\partial a^1} + a^0\frac{\partial}{\partial a^2} + a^1\frac{\partial}{\partial a^3}, \\ E_3 &= a^3\frac{\partial}{\partial a^0} - a^2\frac{\partial}{\partial a^1} - a^1\frac{\partial}{\partial a^2} + a^2\frac{\partial}{\partial a^3}. \end{aligned} \tag{3.14}$$

Computing the commutator relations, we obtain

$$[E_2, E_3] = -2E_1, \quad [E_3, E_1] = 2E_2, \quad [E_1, E_2] = 2E_3. \tag{3.15}$$

Introducing for later use new vector fields

$$X_0 = -\tfrac{1}{2}E_1, \quad X_1 = -\tfrac{1}{2}E_3, \quad X_2 = -\tfrac{1}{2}E_2, \tag{3.16}$$

we obtain

$$[X_1, X_2] = -X_0, \quad [X_2, X_0] = X_1, \quad [X_0, X_1] = X_2. \tag{3.17}$$

If we represent the unit quaternions

$$\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$$

by the matrices

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \tag{3.18}$$

respectively, every Gödel quaternion

$$a^\mu \mathbf{e}_\mu \tag{3.19}$$

goes over to the matrix

$$A = \begin{pmatrix} a^0 + a^3 & a^1 + a^2 \\ -a^1 + a^2 & a^0 - a^3 \end{pmatrix} \tag{3.20}$$

with

$$(a^0)^2 + (a^1)^2 - (a^2)^2 - (a^3)^2 = 1, \tag{3.21}$$

and the quaternion muliplication goes over to the matrix multiplication. This expresses the well-known fact that H^3 is isomorphic to $SLG(2, R)$.

We introduce on H^3 a new coordinate system

$$x^0,\, x^1,\, x^2 \tag{3.22}$$

by the substitutions

$$\begin{aligned} a^0 &= \tfrac{1}{2} e^{\frac{1}{2}x^1} x^2 \cos \tfrac{1}{2}x^0 + \cosh \tfrac{1}{2}x^1 \sin \tfrac{1}{2}x^0, \\ a^1 &= \tfrac{1}{2} e^{\frac{1}{2}x^1} x^2 \sin \tfrac{1}{2}x^0 + \cosh \tfrac{1}{2}x^1 \cos \tfrac{1}{2}x^0, \\ a^2 &= \tfrac{1}{2} e^{\frac{1}{2}x^1} x^2 \sin \tfrac{1}{2}x^0 + \sinh \tfrac{1}{2}x^1 \cos \tfrac{1}{2}x^0, \\ a^3 &= -\tfrac{1}{2} e^{\frac{1}{2}x^1} x^2 \cos \tfrac{1}{2}x^0 + \sinh \tfrac{1}{2}x^1 \sin \tfrac{1}{2}x^0. \end{aligned} \tag{3.23}$$

(This is a two-parametric family of straight lines on H^3—x^0 and x^1 being the parameters—and x^2 is the coordinate along the lines.)

This coordinate system covers H^3 completely. The matrix A is given in this coordinate by

$$A = \begin{pmatrix} e^{\frac{1}{2}x^1} \sin \tfrac{1}{2}x^0 & e^{\frac{1}{2}x^1} \cos \tfrac{1}{2}x^0 \\ e^{\frac{1}{2}x^1} x^2 \sin \tfrac{1}{2}x^0 - e^{-\frac{1}{2}x^1} \cos \tfrac{1}{2}x^0 & e^{\frac{1}{2}x^1} x \cos^2 \tfrac{1}{2}x^0 + e^{-\frac{1}{2}x^1} \sin \tfrac{1}{2}x^0 \end{pmatrix}. \tag{3.24}$$

The left-invariant 1-forms of a matrix group whose general element is given by the matrix A can be obtained by computing

$$\omega = A^{-1}\, dA. \tag{3.25}$$

As shown, for instance, by Flanders,[7] all matrix elements of ω will be left-invariant 1-forms. Carrying out the computation indicated in (3.25), we obtain

$$\omega = \begin{pmatrix} -\tfrac{1}{2}(\cos x^0\, dx^1 + e^{x^1} \sin x^0\, dx^2) & -\tfrac{1}{2}\, dx^0 + \tfrac{1}{2} \sin x^0\, dx^1 - e^{x^1} \cos^2 \tfrac{1}{2}x^0\, dx^2 \\ \tfrac{1}{2}\, dx^0 + \tfrac{1}{2} \sin x^0\, dx^1 + e^{x^1} \sin^2 \tfrac{1}{2}x^0\, dx^2 & \tfrac{1}{2}(\cos x^0\, dx^1 + e^{x^1} \sin x^0\, dx^2) \end{pmatrix}. \tag{3.26}$$

We select from (3.26) the following left-invariant 1-forms:

$$\begin{aligned} \omega^0 &= dx^0 + e^{x^1}\, dx^2, \\ \omega^1 &= \cos x^0\, dx^1 + e^{x^1} \sin x^0\, dx^2, \\ \omega^2 &= -\sin x^0\, dx^1 + e^{x^1} \cos x^0\, dx^2 \end{aligned} \tag{3.27}$$

as the base for the 1-forms on H^3. The corresponding left-invariant vector fields, serving as the base for the vector fields on H^3, are given by

$$\begin{aligned} X_0 &= \frac{\partial}{\partial x^0}, \\ X_1 &= -\sin x^0 \frac{\partial}{\partial x^0} + \cos x^0 \frac{\partial}{\partial x^1} + e^{-x^1} \sin x^0 \frac{\partial}{\partial x^2}, \\ X_2 &= -\cos x^0 \frac{\partial}{\partial x^0} - \sin x^0 \frac{\partial}{\partial x^1} + e^{-x^1} \cos x^0 \frac{\partial}{\partial x^2}. \end{aligned} \tag{3.28}$$

These are the vector fields defined by (3.16), written in the coordinate system (3.22) as defined by the substitutions (3.23), as one can see easily by a straightforward computation.

We now consider the group

$$M_4 = R \times H^3, \tag{3.29}$$

where the coordinate x^3 is introduced on R and

$$X_3 = \frac{\partial}{\partial x^3}. \tag{3.30}$$

Therefore, the left-invariant vector fields on M_4,

$$X_0,\, X_1,\, X_2,\, X_3, \tag{3.31}$$

defined by (3.28) and (3.30), can be chosen for the base of the vector fields on M_4, and the left-invariant 1-forms

$$\omega^0,\, \omega^1,\, \omega^2,\, \omega^3, \tag{3.32}$$

defined by (3.27),and

$$\omega^3 = dx^3 \tag{3.33}$$

are the corresponding base for the 1-forms on M_4. The Lie algebra of the left-invariant vector fields on M_4 is given by

$$\begin{gathered} [X_1, X_2] = -X_0, \quad [X_2, X_0] = X_1, \quad [X_0, X_1] = X_2, \\ [X_a, X_3] = 0, \quad a = 0, 1, 2, \end{gathered} \tag{3.34}$$

or, correspondingly,

$$\begin{aligned} d\omega^0 &= \omega^1 \wedge \omega^2, & d\omega^1 &= -\omega^2 \wedge \omega^0, \\ d\omega^2 &= -\omega^0 \wedge \omega^1, & d\omega^3 &= 0. \end{aligned} \tag{3.35}$$

In the subsequent sections two different pseudo-Riemannian metrics, invariant under the left translations of the group M_4 and satisfying the Einstein equations (2.40), are introduced on M_4, which is by construction simply connected and, therefore, is the uniquely defined universal covering group of the Lie algebra (3.34). These manifolds are called the Class II and Class III universes. We discuss their properties.

4. THE METRIC OF THE CLASS II UNIVERSES

We construct the metric on M_4' as follows. We let

$$R > 0 \quad \text{and} \quad \tfrac{1}{2} < |k| \le (2)^{-\frac{1}{2}} \tag{4.1}$$

be two real parameters and introduce a new basis in the Lie algebra (3.34) by the following substitutions:

$$\begin{aligned} Y_0 &= \frac{1}{R}\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}} X_0 + \frac{1}{R}\left(\frac{2(1-2k^2)}{4k^2-1}\right)^{\frac{1}{2}} X_3, \\ Y_1 &= \frac{2}{R(1-k)^{\frac{1}{2}}} X_1, \quad Y_2 = \frac{2}{R(1+k)^{\frac{1}{2}}} X_2, \\ Y_3 &= \frac{2}{R}\left(\frac{1-2k^2}{4k^2-1}\right)^{\frac{1}{2}} X_0 + \frac{4k^2}{R}\frac{1}{(4k^2-1)^{\frac{1}{2}}} X_3. \end{aligned} \tag{4.2}$$

We define the metric on M_4 by demanding that Y_0, Y_1, Y_2, Y_3 be pseudo-orthonormal, that is,

$$g(Y_a, Y_b) = g_{ab} = \text{diag}\,(+1, -1, -1, -1). \tag{4.3}$$

In other words, we define the line element to be

$$ds^2 = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2, \tag{4.4}$$

where θ^0, θ^1, θ^2, θ^3 is the basis of the 1-forms, corresponding to Y_0, Y_1, Y_2, Y_3 and given by

$$\begin{aligned} \theta^0 &= Rk^2\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}} \omega^0 - R\left(\frac{1-2k^2}{2(4k^2-1)}\right)^{\frac{1}{2}} \omega^3, \\ \theta^1 &= R[\tfrac{1}{2}(1-k)]^{\frac{1}{2}}\omega^1, \quad \theta^2 = R[\tfrac{1}{2}(1+k)]^{\frac{1}{2}}\omega^2, \\ \theta^3 &= -\frac{R}{2}\left(\frac{1-2k^2}{4k^2-1}\right)^{\frac{1}{2}} \omega^0 + \frac{R}{2}\frac{1}{(4k^2-1)^{\frac{1}{2}}}\omega^3. \end{aligned} \tag{4.5}$$

After trivial computations we obtain

$$\begin{aligned} ds^2 = (\tfrac{1}{2}R)^2[&(1+2k^2)(\omega^0)^2 - (1-k)(\omega^1)^2 \\ &- (1+k)(\omega^2)^2 - (\omega^3)^2 - 2(1-2k^2)^{\frac{1}{2}}\omega^0\omega^3], \end{aligned} \tag{4.6}$$

where the ω's satisfy (3.35) and are given in our coordinate system by (3.27). We would like to make the following remarks to (4.6). Since the invariant vector field X_3 commutes with the other vector fields X_a, $a = 0, 1, 2$ [see (3.34)], X_3 is also a generator of M_4. Therefore, in a coordinate system where $X_3 = \partial/\partial x^3$, the ω's do not depend on x^3 [see (3.27) and (3.30)]. But, since X_3 is not hypersurface orthogonal, we cannot get rid of the "cross terms" in the metric. Consider now the vector field $K = \partial/\partial x^2$. One sees that $[K, X_a] = 0$, $a = 0, 1, 2, 3$; and that therefore K is also a generator of M_4; consequently, the ω's and the metric are independent of x^2. Since

$$g(K, K) = (\tfrac{1}{2}R)^2 k(2k - \cos 2x^0)e^{2x^1} > 0$$

for $\frac{1}{2} < |k|$, K is a timelike generator of M_4 and x^2 a timelike coordinate. Therefore, (4.6) in our coordinate system exhibits the fact that the metric is stationary. But it is not static, since there is no hypersurface orthogonal time like Killing vector field.

It will turn out that the vector field Y_0 is tangent to the world lines of the matter. We introduce now new coordinates

$$\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \tag{4.7}$$

by the substitutions

$$\begin{aligned} \tilde{x}^0 = R[\tfrac{1}{2}(4k^2-1)]^{\frac{1}{2}}x^0, \quad \tilde{x}^1 = x^1, \quad \tilde{x}^2 = x^2, \\ \tilde{x}^3 = -(1-2k^2)^{\frac{1}{2}}x^0 + x^3 \end{aligned} \tag{4.8}$$

or

$$\begin{aligned} x^0 = \frac{1}{R}\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}}\tilde{x}^0, \quad x^1 = \tilde{x}^1, \quad x^2 = \tilde{x}^2, \\ x^3 = \frac{1}{R}\left(\frac{2(1-2k^2)}{4k^2-1}\right)^{\frac{1}{2}}\tilde{x}^0 + \tilde{x}^3. \end{aligned} \tag{4.9}$$

We see that

$$Y = \frac{\partial}{\partial \tilde{x}^0}, \tag{4.10}$$

which shows that the matter is at rest with respect to the coordinates (4.7).

Carrying out straightforward calculations, we find that

$$\begin{aligned} \omega^0 &= \frac{1}{R}\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}} d\tilde{x}^0 + \exp(\tilde{x}^1)\, d\tilde{x}^2, \\ \omega^1 &= \cos\frac{1}{R}\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}}\tilde{x}^0\, d\tilde{x}^1 \\ &\quad + \exp(\tilde{x}^1)\sin\frac{1}{R}\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}}\tilde{x}^0\, d\tilde{x}^2, \\ \omega^2 &= -\sin\frac{1}{R}\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}}\tilde{x}^0\, dx^1 \\ &\quad + \exp(\tilde{x}^1)\cos\frac{1}{R}\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}}\tilde{x}^0\, d\tilde{x}^2, \\ \omega^3 &= \frac{1}{R}\left(\frac{2(1-2k^2)}{4k^2-1}\right)^{\frac{1}{2}} d\tilde{x}^0 + d\tilde{x}^3. \end{aligned} \tag{4.11}$$

Substituting (4.11) into (4.6), we obtain the metric in the new coordinate system (4.7). Carrying out these computations, we see that the metric has the form

$$ds^2 = (d\tilde{x}^0)^2 + 2\tilde{p}_\alpha\tilde{\omega}^\alpha\, d\tilde{x}^0 + \tilde{g}_{\alpha\beta}\tilde{\omega}^\alpha\tilde{\omega}^\beta, \qquad \alpha, \beta, \gamma, \cdots = 1, 2, 3, \tag{4.12}$$

where $\tilde{p}_\alpha$ and $\tilde{g}_{\alpha\beta}$ are functions of $\tilde{x}^0$ alone and $\tilde{\omega}^1$, $\tilde{\omega}^2$, $\tilde{\omega}^3$ are 1-forms given by

$$\tilde{p}_\alpha = \left(0, Rk^2\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}}, R\left(\frac{1-2k^2}{2(4k^2-1)}\right)^{\frac{1}{2}}\right), \tag{4.13}$$

$$\tilde{g}_{\alpha\beta} = (\tfrac{1}{2}R)^2\begin{pmatrix} -1+k\cos 2\tilde{x}^0 & k\sin 2\tilde{x}^0 & 0 \\ k\sin 2\tilde{x}^0 & k(2k-\cos 2\tilde{x}^0) & -(1-2k^2)^{\frac{1}{2}} \\ 0 & -(1-2k^2)^{\frac{1}{2}} & -1 \end{pmatrix}, \tag{4.14}$$

and

$$\tilde{\omega}^1 = d\tilde{x}^1, \quad \tilde{\omega}^2 = \exp(\tilde{x}^1)\, d\tilde{x}^2, \quad \tilde{\omega}^3 = d\tilde{x}^3, \tag{4.15}$$

respectively. The 1-forms (4.15) are the left-invariant 1-forms of the group of Bianchi type III since

$$d\tilde{\omega}^1 = 0, \quad d\tilde{\omega}^2 = \tilde{\omega}^1 \wedge \tilde{\omega}^2, \quad d\tilde{\omega}^3 = 0, \tag{4.16}$$

as one sees immediately. Our solution is, therefore, a special case of spatially homogeneous solutions admitting the group (4.16).

We introduce now another coordinate system

$$\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3 \tag{4.17}$$

by the substitutions

$$\bar{x}^0 = x^0 - \frac{(1-2k^2)^{\frac{1}{2}}}{1+2k^2}x^3, \quad \bar{x}^1 = x^1, \quad \bar{x}^2 = x^2,$$
$$\bar{x}^3 = \left(\frac{2}{1+2k^2}\right)^{\frac{1}{2}} x^3 \tag{4.18}$$

or

$$x^0 = \bar{x}^0 + \left(\frac{1-2k^2}{2(1+2k^2)}\right)^{\frac{1}{2}}\bar{x}^3, \quad \bar{x}^1 = x^1, \quad \bar{x}^2 = x^2,$$
$$x^3 = \tfrac{1}{2}[(1+2k^2)]^{\frac{1}{2}}\bar{x}^3. \tag{4.19}$$

We will see that $\bar{x}^3 = \text{const}$ are the H^3 hypersurfaces.

Carrying out these coordinate transformations, we obtain from (4.2) the following expressions:

$$Y_0 = \frac{1}{R}\frac{4k^2}{1+2k^2}\left(\frac{2}{4k^2-1}\right)^{\frac{1}{2}}\bar{X}_0 + \frac{2}{R}\left(\frac{1-2k^2}{(4k^2-1)(1+2k^2)}\right)^{\frac{1}{2}}\bar{X}_3,$$
$$Y_1 = \frac{2}{R(1-k)^{\frac{1}{2}}}(\bar{X}_1\cos\beta\bar{x}^3 + \bar{X}_3\sin\beta\bar{x}^3),$$
$$Y_2 = \frac{2}{R(1+k)^{\frac{1}{2}}}(-\bar{X}_1\sin\beta\bar{x}^3 + \bar{X}_3\cos\beta\bar{x}^3), \tag{4.20}$$
$$Y_3 = \frac{2}{R}\frac{1}{1+2k^2}\left(\frac{1-2k^2}{4k^2-1}\right)^{\frac{1}{2}}\bar{X}_0 + \frac{4k^2}{R}\left(\frac{2}{(4k^2-1)(1+2k^2)}\right)^{\frac{1}{2}}\bar{X}_3,$$

where

$$\beta = [(1-2k^2)/2(1+2k^2)]^{\frac{1}{2}} \tag{4.21}$$

and

$$\bar{X}_0 = \frac{\partial}{\partial\bar{x}^0},$$
$$\bar{X}_1 = -\sin\bar{x}^0\frac{\partial}{\partial\bar{x}^0} + \cos\bar{x}^0\frac{\partial}{\partial\bar{x}^1} + \exp(-\bar{x}^1)\sin\bar{x}^0\frac{\partial}{\partial\bar{x}^2},$$
$$\bar{X}_2 = -\cos\bar{x}^0\frac{\partial}{\partial\bar{x}^0} - \sin\bar{x}^0\frac{\partial}{\partial\bar{x}^1} + \exp(-\bar{x}^1)\cos\bar{x}^0\frac{\partial}{\partial\bar{x}^2}, \tag{4.22}$$
$$\bar{X}_3 = \frac{\partial}{\partial\bar{x}^3}.$$

We now introduce a new basis for the Lie algebra of the left-invariant vector fields on M_4 by the following substitutions:

$$Z_0 = 2k^2\left(\frac{2}{(4k^2-1)(1+2k^2)}\right)^{\frac{1}{2}}Y_0 - \left(\frac{1-2k^2}{(4k^2-1)(1+2k^2)}\right)^{\frac{1}{2}}Y_3,$$
$$Z_1 = Y_1, \quad Z_2 = Y_2, \tag{4.23}$$
$$Z_3 = -\left(\frac{1-2k^2}{(4k^2-1)(1+2k^2)}\right)^{\frac{1}{2}}Y_0 + 2k^2\left(\frac{2}{(4k^2-1)(1+2k^2)}\right)^{\frac{1}{2}}Y_3.$$

Since (4.23) is a Lorentz transformation between the Z's and the Y's, the metrics defined by

$$g(Z_a, Z_b) = g_{ab} = \operatorname{diag}(+1, -1, -1, -1) \tag{4.24}$$

and

$$g(Y_a, Y_b) = g_{ab} = \operatorname{diag}(+1, -1, -1, -1) \tag{4.25}$$

are identical. After obvious substitutions we obtain

$$
\begin{aligned}
Z_0 &= \frac{2}{R(1+2k^2)^{\frac12}}\,\bar{X}_0,\\
Z_1 &= \frac{2}{R(1-k)^{\frac12}}\,(\bar{X}_1\cos\beta\bar{x}^3+\bar{X}_2\sin\beta\bar{x}^3),\\
Z_2 &= \frac{2}{R(1+k)^{\frac12}}\,(-\bar{X}_1\sin\beta\bar{x}^3+\bar{X}_2\cos\beta\bar{x}^3),\\
Z_3 &= \frac{2}{R}\,\bar{X}_3,
\end{aligned}
\tag{4.26}
$$

where β and $\bar{X}_0$, $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$ are given by (4.21) and (4.22), respectively. The corresponding left-invariant 1-forms are

$$
\begin{aligned}
\phi^0 &= \tfrac12 R(1+2k^2)^{\frac12}\bar{\omega}^0,\\
\phi^1 &= \tfrac12 R(1-k)^{\frac12}(\bar{\omega}^1\cos\beta\bar{x}^3+\bar{\omega}^2\sin\beta\bar{x}^3),\\
\phi^2 &= \tfrac12 R(1+k)^{\frac12}(-\bar{\omega}^1\sin\beta\bar{x}^3+\bar{\omega}^2\cos\beta\bar{x}^3),\\
\phi^3 &= \tfrac12 R\bar{\omega}^3.
\end{aligned}
\tag{4.27}
$$

As the consequence of all that, the line element

$$
\begin{aligned}
ds^2 = {}& (\tfrac12 R)^2(1+2k^2)(\bar{\omega}^0)^2\\
&-(1-k)(\bar{\omega}^1\cos\beta\bar{x}^3+\bar{\omega}^2\sin\beta\bar{x}^3)^2\\
&-(1+k)(-\bar{\omega}^1\sin\beta\bar{x}^3+\bar{\omega}^2\cos\beta\bar{x}^3)^2-(\bar{\omega}^3)^2
\end{aligned}
\tag{4.28}
$$

is the same as (4.6) but is in the new coordinate system, where $\bar{\omega}^0$, $\bar{\omega}^1$, $\bar{\omega}^2$, $\bar{\omega}^3$ are given by

$$
\begin{aligned}
\bar{\omega}^0 &= d\bar{x}^0+\exp(\bar{x}^1)\,d\bar{x}^2,\\
\bar{\omega}^1 &= \cos\bar{x}^0\,d\bar{x}^1+\exp\bar{x}^1\sin\bar{x}^0\,d\bar{x}^2,\\
\bar{\omega}^2 &= -\sin\bar{x}^0\,d\bar{x}^1+\exp(\bar{x}^1)\cos\bar{x}^0\,d\bar{x}^2,\\
\bar{\omega}^3 &= d\bar{x}^3.
\end{aligned}
\tag{4.29}
$$

To sum up our findings, we see that we used two different bases for the Lie algebra of the left-invariant vector fields on M_4, namely,

$$Y_0,\, Y_1,\, Y_2,\, Y_3 \tag{4.30}$$

and

$$Z_0, Z_1, Z_2, Z_3. \tag{4.31}$$

We shall see in the next section that (4.30) is intimately connected with the motion of the matter in our solutions. (4.31) is distinguished by the geometry of the 3-dimensional hypersurfaces corresponding to the normal subgroup H^3 of M_4, as we shall see in Sec. 6.

The three different coordinate systems employed differ as follows: In (4.6) with the coordinates x^0, x^1, x^2, x^3, the x^2 lines are the integral curves of the time-like generators of M_4; in (4.12) the $\tilde{x}^0$ lines are the world lines of the matter, as we shall see in the next section; in (4.28) the $\bar{x}^3$ lines are perpendicular to the 3-dimensional hypersurfaces corresponding to the normal subgroup. In case of $k=\frac12$ we obtain a cosmos filled with radiation.

5. MISCELLANEOUS RESULTS AND THE MOTION OF THE MATTER

Consider the basis Y_0, Y_1, Y_2, Y_3 defined by (4.2). The Lie algebra of M_4 is given in this basis by the following commutation relations:

$$
\begin{aligned}
[Y_1, Y_2] &= -\frac{4k^2}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac12} Y_0 + \frac{2}{R}\left(\frac{1-2k^2}{(1-k^2)(4k^2-1)}\right)^{\frac12} Y_3,\\
[Y_2, Y_0] &= \frac{1-k}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac12} Y_1,\\
[Y_0, Y_1] &= \frac{1+k}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac12} Y_2,\\
[Y_0, Y_3] &= 0,\\
[Y_1, Y_3] &= -\frac{2(1+k)}{R}\left(\frac{1-2k^2}{(1-k^2)(4k^2-1)}\right)^{\frac12} Y_2,\\
[Y_2, Y_3] &= +\frac{2(1-k)}{R}\left(\frac{1-2k^2}{(1-k^2)(4k^2-1)}\right)^{\frac12} Y_1.
\end{aligned}
\tag{5.1}
$$

Using (2.32), we compute the components of the affine connection:

$$
\begin{aligned}
\Gamma_{012} &= -\frac{1-2k^2}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac12},\\
\Gamma_{123} &= -\frac{1+2k}{R}\left(\frac{1-2k^2}{(1-k^2)(4k^2-1)}\right)^{\frac12},\\
\Gamma_{120} &= -\frac{k(1+2k)}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac12},\\
\Gamma_{231} &= -\frac{1-2k}{R}\left(\frac{1-2k^2}{(1-k^2)(4k^2-1)}\right)^{\frac12},\\
\Gamma_{210} &= -\frac{k(1-2k)}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac12},\\
\Gamma_{312} &= -\frac{1}{R}\left(\frac{1-2k^2}{(1-k^2)(4k^2-1)}\right)^{\frac12}.
\end{aligned}
\tag{5.2}
$$

Using (2.37), we see that the components of the Ricci tensor field are given by

$$
\begin{aligned}
R_{ab} = \operatorname{diag}\biggl(&-\frac{4k^2}{R^2(1-k^2)},\ \frac{2(1-2k^2)}{R^2(1-k^2)},\\
&\frac{2(1-2k^2)}{R^2(1-k^2)},\ \frac{2(1-2k^2)}{R^2(1-k^2)}\biggr).
\end{aligned}
\tag{5.3}
$$

Comparing with (2.40), we see that we indeed have a solution of the field equations, with

$$u_a = (1, 0, 0, 0) \tag{5.4}$$

and

$$\frac{\kappa\rho}{2\Lambda} = -(4k^2 - 1), \quad \Lambda = -\frac{1}{R^2(1 - k^2)}. \tag{5.5}$$

The meaning of (5.4) is that Y_0 is the velocity vector field of the matter. Since $Y_0 = \partial/\partial\tilde{x}^0$ [see (4.10)], it follows that in (4.12) the $\tilde{x}^0$ lines are the world lines of the matter, as stated earlier.

In order to investigate the motion of the matter, we have to integrate the equations[8]

$$L_{Y_0}(Y) = \nabla_{Y_0}(Y) - \nabla_Y(Y_0), \tag{5.6}$$

where Y is perpendicular to Y_0, that is,

$$Y = \eta^a Y_a, \tag{5.7}$$

the summation extending over 1, 2, 3.

The vector Y is a vector perpendicular to a particle geodesic, and the tip of its arrow is in the neighboring particle geodesic.

Substituting (5.7) into (5.6), applying the rules of the covariant derivation, and introducing the notation

$$\dot{\eta}^a = Y_0\eta^a,$$

we obtain the equations

$$\dot{\eta}^a = C^a{}_{b0}\eta^b. \tag{5.8}$$

Using (5.1), we get

$$\begin{aligned}
\dot{\eta}^1 &= \frac{1-k}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}\eta^2,\\
\dot{\eta} &= -\frac{1+k}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}\eta^1,\\
\dot{\eta}^3 &= 0.
\end{aligned} \tag{5.9}$$

These equations describe the motion of the matter with respect to the 3-dimensional vector frame of the Y_a. As a consequence of these equations, we have

$$(1 + k)\dot{\eta}^1\eta^1 + (1 - k)\dot{\eta}^2\eta^2 = 0,$$

and by integration we obtain

$$[\eta^1/(1 - k)^{\frac{1}{2}}]^2 + [\eta^2/(1 + k)^{\frac{1}{2}}]^2 = A^2, \quad \eta^3 = B, \tag{5.10}$$

as the equation of the orbit for the neighboring particle. The orbits of the particles in the Y_a frame are, therefore, ellipses in the (Y_1, Y_2) plane. The main axes of the ellipses are in the Y_1 and Y_2 directions. The axes of the ellipse rotate around Y_3 with respect to the inertial compass. To see that, we determine the motion of the frame Y_a along the world lines of the matter. Using the formula

$$\dot{Y}_a \equiv \nabla_{Y_0}(Y_a) = \Gamma_{0a}{}^b\, Y_b$$

and (5.2), we obtain the equations

$$\begin{aligned}
\dot{Y}_0 &= 0,\\
\dot{Y}_1 &= \frac{1-2k^2}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}Y_2,\\
\dot{Y}_2 &= -\frac{1-2k^2}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}Y_1,\\
\dot{Y}_3 &= 0.
\end{aligned} \tag{5.11}$$

The content of these equations is that Y^3 is parallel propagated along the $\tilde{x}^0$ lines and Y_1 and Y_2 and that the axes of the ellipse (5.10) are rotating around Y_3 with respect to the parallel propagated frame.[9] The angular velocity of this rotation is given by

$$\omega_{Y\ \text{frame}} = \frac{1-2k^2}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}.$$

This gives a characterization for the frame Y_0, Y_1, Y_2, Y_3 by the motion of the matter.

Another way to bring Y_1, Y_2, Y_3 in connection with the motion of the matter is to decompose the tensor field

$$U(X, Z) = (\nabla_X\theta^0)(Z) = X\theta^0(Z) - \theta^0(\nabla_X(Z))$$

into symmetric and skew-symmetric parts (θ^0 is the covariant tensor field, 1-form, corresponding to Y_0). The components of the tensor field U are given by

$$U_{ab} = U(Y_a, T_b) = -\Gamma_{ab}{}^0 \quad \text{[see (2.9) and (2.10)]}.$$

The symmetric part, the tensor of shear σ, has the nonvanishing components

$$\sigma_{12} = \sigma_{21} = \frac{k}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}. \tag{5.12}$$

The skew-symmetric part, the tensor of rotation w, has the nonvanishing components

$$w_{12} = -w_{21} = \frac{2k^2}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}. \tag{5.13}$$

The nonvanishing component of the rotation vector V, defined by

$$v^a = -\tfrac{1}{2}\eta^{abcd}u_a w_{cd}, \tag{5.14}$$

is given by

$$v^3 = \frac{2k^2}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}. \tag{5.15}$$

Therefore, writing the tensor fields in contravariant form, we obtain

$$\sigma = \frac{k}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}(Y_1 \otimes Y_2 + Y_2 \otimes Y_1) \tag{5.16}$$

or

$$\sigma = \frac{k}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}}[2^{-\frac{1}{2}}(Y_1+Y_2) \otimes 2^{-\frac{1}{2}}(Y_1+Y_2) - 2^{-\frac{1}{2}}(Y_1-Y_2) \otimes 2^{-\frac{1}{2}}(Y_1-Y_2)]; \quad (5.17)$$

that is, the eigenvalues of σ are

$$\pm\frac{k}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}} \quad (5.18)$$

and the corresponding eigenvectors are

$$2^{-\frac{1}{2}}(Y_1 \pm Y_2). \quad (5.19)$$

The vector of rotation is given by

$$V = \frac{2k^2}{R}\left(\frac{2}{(1-k^2)(4k^2-1)}\right)^{\frac{1}{2}} Y_3. \quad (5.20)$$

For no value of k, $\frac{1}{2} < |k| \le 2^{-\frac{1}{2}}$, is the shear or rotation vanishing. The Gödel cosmos is therefore not contained in Class II. This concludes the characterization of the frame Y_0, Y_1, Y_2, Y_3 by the motion of the matter.

Using (2.36), we can calculate the components of the curvature tensor field and the Weyl tensor field C, which turns out to be of Type I and can be given by the nonvanishing components

$$\begin{aligned}
C_{2323} &= -C_{1010} = \frac{2k^2}{3R^2(1-k^2)},\\
C_{3131} &= -C_{2020} = \frac{2k^2}{3R^2(1-k^2)},\\
C_{1212} &= -C_{3030} = -\frac{4k^2}{3R^2(1-k^2)},\\
C_{2310} &= -C_{3120} = \frac{2k}{R^2(1-k^2)}[\tfrac{1}{2}(1-2k^2)]^{\frac{1}{2}},
\end{aligned} \quad (5.21)$$

defined by

$$C_{abcd} = C(W_a, W_b, W_c, W_d), \quad (5.22)$$

where

$$\begin{aligned}
W_0 &= \frac{1}{(4k^2-1)^{\frac{1}{2}}} Y_0 - \left(\frac{2(1-2k^2)}{4k^2-1}\right)^{\frac{1}{2}},\\
W_1 &= 2^{-\frac{1}{2}}(Y_1 - Y_2), \quad W_2 = 2^{-\frac{1}{2}}(Y_1+Y_2),\\
W_3 &= -\left(\frac{2(1-2k^2)}{4k^2-1}\right)^{\frac{1}{2}} Y_0 + \frac{1}{(4k^2-1)^{\frac{1}{2}}} Y_3.
\end{aligned} \quad (5.23)$$

We notice that the Weyl vector fields W_0, W_1, W_2, and W_3 can be regarded as the eigenvector fields of the tensor of shear [see (5.13)]. W_0 and W_3 belong to the eigenvalue zero.

6. GEOMETRY OF THE SOLUTION

Consider the basis Z_0, Z_1, Z_2, Z_3 defined by (4.26). The Lie algebra of M_4 is given in this basis by the following commutation relations:

$$\begin{aligned}
[Z_1, Z_2] &= -\frac{2}{R}\frac{1+2k^2}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}} Z_0,\\
[Z_2, Z_0] &= \frac{2}{R}\frac{1-k}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}} Z_1,\\
[Z_0, Z_1] &= \frac{2}{R}\frac{1+k}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}} Z_2,\\
[Z_0, Z_3] &= 0,\\
[Z_1, Z_3] &= -\frac{1+k}{R}\left(\frac{2(1-2k^2)}{(1-k^2)(1+2k^2)}\right)^{\frac{1}{2}} Z_2,\\
[Z_2, Z_3] &= +\frac{1-k}{R}\left(\frac{2(1-2k^2)}{(1-k^2)(1+2k^2)}\right)^{\frac{1}{2}} Z_1.
\end{aligned} \quad (6.1)$$

The first three commutation relations show that Z_0, Z_1, Z_2 form together the basis of a 3-dimensional subalgebra of the Lie algebra of M_4. The second three commutation relations indicate that the subalgebra is an ideal. This ideal generates H^3.

Using (2.32), we compute the components of the affine connection

$$\begin{aligned}
\Gamma_{012} &= -\frac{1}{R}\frac{1-2k^2}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}},\\
\Gamma_{123} &= -\frac{k}{R}\left(\frac{2(1-2k^2)}{(1-k^2)(1+2k^2)}\right)^{\frac{1}{2}},\\
\Gamma_{120} &= -\frac{1}{R}\frac{1+2k+2k^2}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}},\\
\Gamma_{231} &= \frac{k}{R}\left(\frac{2(1-2k^2)}{(1-k^2)(1+2k^2)}\right)^{\frac{1}{2}},\\
\Gamma_{210} &= \frac{1}{R}\frac{1-2k+2k^2}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}},\\
\Gamma_{312} &= -\frac{1}{R}\left(\frac{2(1-2k^2)}{(1-k^2)(1+2k^2)}\right)^{\frac{1}{2}}.
\end{aligned} \quad (6.2)$$

From (6.2) we can read out a bit of geometry. Since

$$\nabla_{Z_a}(Z_a) = \Gamma_{aa}{}^{b} Z_b = 0, \quad a = 0, 1, 2, 3,$$

it follows that the vector fields Z_0, Z_1, $\dot{Z}_2$, Z_3 are geodesic. Denoting $\nabla_{Z_3}(Z_a)$ by $\dot{Z}_a$, we have

$$\begin{aligned}
\dot{Z}_0 &= 0,\\
\dot{Z}_1 &= \frac{1}{R}\left(\frac{2(1-2k^2)}{(1-k^2)(1+2k^2)}\right)^{\frac{1}{2}} Z_2,\\
\dot{Z}_2 &= -\left(\frac{2(1-2k^2)}{(1-k^2)(1+2k^2)}\right)^{\frac{1}{2}} Z_1,\\
Z_3 &= 0.
\end{aligned} \quad (6.3)$$

Therefore, it follows that Z_0 is parallel-propagated along the $\bar{x}^3$ lines and Z_1 and Z_2 rotate around Z_0 with respect to the parallel-propagated frame. What is the geometrical meaning of Z_0, Z_1, Z_2? These three vector fields are tangential to the 3-spaces $\bar{x}^3 = \text{const}$. Looking into the geometry of these 3-spaces, we see that the components of the affine connection and the Ricci tensor field are given by

$$\begin{aligned}
\Gamma_{012} &= -\frac{1}{R}\frac{1-2k^2}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}}, \\
\Gamma_{120} &= -\frac{1}{R}\frac{1+2k+2k^2}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}}, \qquad (6.4) \\
\Gamma_{201} &= -\frac{1}{R}\frac{1-2k+2k^2}{[(1-k^2)(1+2k^2)]^{\frac{1}{2}}},
\end{aligned}$$

and

$$R_{ab} = \operatorname{diag}\left(-\frac{2(1+2k+2k^2)(1-2k+2k^2)}{R^2(1-k^2)(1+2k^2)}, \frac{2(1-2k^2)(1-2k+2k^2)}{R^2(1-k^2)(1+2k^2)}, \frac{2(1-2k^2)(1+2k+2k^2)}{R^2(1-k^2)(1+2k^2)}\right), \qquad (6.5)$$

respectively; therefore, Z_0, Z_1, Z_2 are the eigenvector fields of the Ricci tensor field. The eigenvalues do not depend on $\bar{x}^3$ [see (6.5)]. The nonvanishing components of the curvature tensor field are

$$\begin{aligned}
R_{1212} &= \frac{(1-12k^4)}{R^2(1-k^2)(1+2k^2)}, \\
R_{2020} &= -\frac{(1+2k-2k^2)^2}{R^2(1-k^2)(1+2k^2)}, \qquad (6.6) \\
R_{0101} &= -\frac{(1-2k-2k^2)^2}{R^2(1-k^2)(1+2k^2)}
\end{aligned}$$

[see (2.36)]. These are also independent of $\bar{x}^3$.

Introducing the notations

$$\begin{aligned}
\rho_0 &= 1 - 12k^4, \\
\rho_1 &= -(1+2k-2k^2)^2 \\
&= -4[k-\tfrac{1}{2}(1+\sqrt{3})]^2[k-\tfrac{1}{2}(1-\sqrt{3})]^2, \qquad (6.7) \\
\rho_2 &= -(1-2k-2k^2)^2 \\
&= -4[k+\tfrac{1}{2}(1+\sqrt{3})]^2[k+\tfrac{1}{2}(1-\sqrt{3})]^2
\end{aligned}$$

and constructing Fig. 1, we obtain an impression about the dependence of (6.6) on the parameter k. Changing the sign of k is equivalent to changing the one and two directions. ρ_0 remains unchanged under the switch of sign in k.

The geometric meaning of (6.6) is as follows: R_{1212}, R_{2020}, R_{0101} are the Gaussian curvatures of the geodesic surfaces spanned by the vectors Z_1Z_2, Z_2Z_0, and Z_0Z_1, respectively, at the point in question.

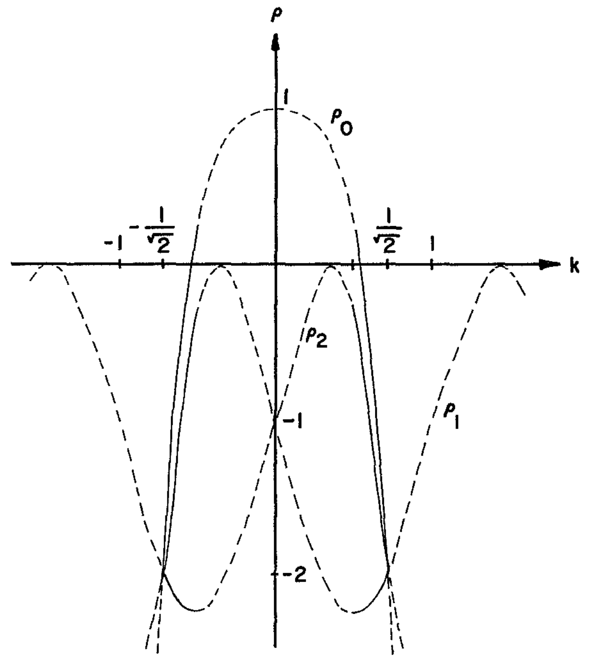

FIG. 1. ρ_0, ρ_1, ρ_2 are proportional to the Gaussian curvature of the geodesic 2-surfaces spanned by Z_1Z_2, Z_2Z_0, and Z_0Z_1, respectively.

From this it follows that the geometry of the hypersurfaces

$$\begin{aligned}
ds^2 = (\tfrac{1}{2}R)^2[(1+2k^2)(\bar{\omega}^0)^2 &- (1-k) \\
\times (\bar{\omega}^1 \cos\beta\bar{x}^3 + \bar{\omega}^2 \sin\beta\bar{x}^3)^2 &- (1+k) \\
-(-\bar{\omega}^1 \sin\beta\bar{x}^3 &+ \bar{\omega}^2\cos\beta\bar{x}^3)^2]
\end{aligned}$$

[see (4.28)] is independent of $\bar{x}^3$ and is given by the geometry of the space

$$ds^2 = (\tfrac{1}{2}R)^2[(1+2k^2)(\bar{\omega}^0)^2 - (1-k)(\bar{\omega}^1)^2 - (1+k)(\bar{\omega}^2)^2]. \qquad (6.8)$$

We can think of the space–time (4.28) as a 1-parametric family of 3-dimensional hypersurfaces—$\bar{x}^3$ being the parameter. These hypersurfaces are generated by H^3 and all have the geometry of (6.8). The $\bar{x}^3$ lines are perpendicular to these hypersurfaces, which are embedded in (4.28) such that the Z_1 and Z_2 directions rotate around Z_0 as we move along the $\bar{x}^3$ lines. This is the geometrical content of (6.3). The $\bar{x}^3$ lines are spacelike; therefore, no physical observer can actually move along them.

It is probably interesting to point out the difference, or similarity between the Class I[1] and the Class II universes. We can think of the Class I universes as a 1-parametric family of 3-dimensional hypersurfaces—$\bar{t}$ being the parameter. These hypersurfaces are generated by S^3, and all have the geometry of

$$ds^2 = -(\tfrac{1}{2}R)^2[(1-k)(\bar{\omega}^1)^2 + (1+k)(\bar{\omega}^2)^2 + (1+2k^2)(\bar{\omega}^3)^2].$$

(The range of k is $|k| < \frac{1}{2}$ and

$$d\bar{\omega}^1 = -\bar{\omega}^2 \wedge \bar{\omega}^3,$$
$$d\bar{\omega}^2 = -\bar{\omega}^3 \wedge \bar{\omega}^1,$$
$$d\bar{\omega}^3 = -\bar{\omega}^1 \wedge \bar{\omega}^2.)$$

The $\bar{t}$ lines are perpendicular to these hypersurfaces, which are embedded in the Class I universes such that the Z_1 and Z_2 directions rotate around Z_3 as we move along the $\bar{t}$ lines. The $\bar{t}$ lines are timelike; therefore, physical observers can actually move along them.

We return now to the discussion of the Class II universes. One sees from Fig. 1 that the geometry of (6.8) is very simple at

$$k = 2^{-\frac{1}{2}}.$$

Our metric is then a special case of Class III universes, as we shall see later.

In order to verify our observation at the end of Sec. 4, we compute the components of the Ricci tensor field with respect to the Z's. Using (2.37) and (6.2), we obtain the following nonvanishing components:

$$\begin{aligned} R_{00} &= -\frac{2}{R^2}\frac{1+4k^4}{(1-k^2)(1+2k^2)}, \\ R_{03} &= \frac{4k^2}{R^2}\frac{[2(1-2k^2)]^{\frac{1}{2}}}{(1-k^2)(1+2k^2)}, \\ R_{11} &= R_{22} = \frac{2}{R^2}\frac{1-2k^2}{1-k^2}, \\ R_{33} &= \frac{4k^2}{R^2}\frac{1-2k^2}{(1-k^2)(1+2k^2)}. \end{aligned} \tag{6.9}$$

In the case of $k = \frac{1}{2}$, (6.9) takes the form

$$R_{00} = -20/9R^2, \qquad R_{03} = 8/9R^2,$$
$$R_{11} = R_{22} = 12/9R^2, \quad R_{33} = 4/9R^2. \tag{6.10}$$

Using (2.40), one sees that the field equations can be satisfied by

$$u_a = (1, 0, 0, -1),$$
$$\kappa\rho = 8/9R^2, \quad \Lambda = -16/9R^2. \tag{6.11}$$

Since $u^a u_a = 0$, one can interpret this model as filled with radiation having the energy density ρ and a Λ term. With these remarks, we close our discussions of the Class II universes.

7. THE METRIC OF THE CLASS III UNIVERSES

We consider the Lie group $R \times H^3$ as before and impose on it another metric as follows. Let

$$\rho > 0 \quad \text{and} \quad |s| < 1 \tag{7.1}$$

be two real parameters and introduce in the Lie algebra (3.34) a new basis by

$$\begin{aligned} Y_0 &= (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}} X_0, \\ Y_1 &= (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}}[2/(1+s)]^{\frac{1}{2}} X_1, \\ Y_2 &= (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}}[2/(1-s)]^{\frac{1}{2}} X_2, \\ Y_3 &= (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}} X_3, \end{aligned} \tag{7.2}$$

and we define the metric on M_4 by demanding that Y_0, Y_1, Y_2, Y_3 be pseudo-orthonormal, that is, that

$$g(Y_a, Y_b) = g_{ab} = \operatorname{diag}(+1, -1, -1, -1). \tag{7.3}$$

In other words, we define the line element to be

$$ds^2 = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2, \tag{7.4}$$

where θ^0, θ^1, θ^2, θ^3 form the corresponding basis of the left-invariant 1-forms, that is,

$$\begin{aligned} \theta^0 &= (2/\kappa\rho)^{\frac{1}{2}}\omega^0, \\ \theta^1 &= (2/\kappa\rho)^{\frac{1}{2}}[\tfrac{1}{2}(1+s)]^{\frac{1}{2}}\omega^1, \\ \theta^2 &= (2/\kappa\rho)^{\frac{1}{2}}[\tfrac{1}{2}(1-s)]^{\frac{1}{2}}\omega^2, \\ \theta^3 &= (2/\kappa\rho)^{\frac{1}{2}}\omega^3. \end{aligned} \tag{7.5}$$

After trivial substitutions, we obtain

$$\begin{aligned} ds^2 = (2/\kappa\rho)[(\omega^0)^2 &- \tfrac{1}{2}(1+s)(\omega^1)^2 \\ &- \tfrac{1}{2}(1-s)(\omega^2)^2 - (\omega^3)^2], \end{aligned} \tag{7.6}$$

where the ω's satisfy (3.35) and, if we use the coordinate system introduced before, can be given by (3.27) and (3.33). All the physical and geometrical investigation can be carried out in the frame of the Y's given by (7.2) and in the coordinate system introduced in Sec. 3, since the two different frames and the three different coordinate systems introduced in the case of the Class II universes coincide here, due to the simplicity of the line element (7.6).

At $s = 0$ we have the Gödel cosmos as we see in Sec. 8.

8. MISCELLANEOUS RESULTS AND THE MOTION OF THE MATTER

Consider the basis Y_0, Y_1, Y_2, Y_3 defined by (7.2). The Lie algebra of M_4 is given in this basis by the following commutation relations:

$$\begin{aligned} [Y_1, Y_2] &= -(\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}}\frac{2}{(1-s^2)^{\frac{1}{2}}} Y_0, \\ [Y_2, Y_0] &= (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}}\frac{1+s}{(1-s^2)^{\frac{1}{2}}} Y_1, \\ [Y_0, Y_1] &= (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}}\frac{1-s}{(1-s^2)^{\frac{1}{2}}} Y_2, \\ [Y_a, Y_3] &= 0, \quad a = 0, 1, 2. \end{aligned} \tag{8.1}$$

Using (2.32), we compute the components of the affine connection

$$\Gamma_{012} = 0,$$
$$\Gamma_{120} = -(\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}} \frac{1-s}{(1-s^2)^{\frac{1}{2}}}, \qquad (8.2)$$
$$\Gamma_{201} = -(\tfrac{1}{2}\chi\rho)^{\frac{1}{2}} \frac{1+s}{(1-s^2)^{\frac{1}{2}}}.$$

Using (2.37), we find that the only nonvanishing component of the Ricci tensor field is given by

$$R_{00} = -\kappa\rho. \qquad (8.3)$$

Comparing this with (2.40), we see that

$$\Lambda = -\tfrac{1}{2}\kappa\rho \qquad (8.4)$$

and

$$u_a = (1, 0, 0, 0). \qquad (8.5)$$

The meaning of (8.5) is that Y_0 is the velocity vector field of the matter. Since $Y_0 = (\frac{1}{2}\kappa\rho)^{\frac{1}{2}}\partial/\partial x^0$, the $t = (2/\kappa\rho)^{\frac{1}{2}}x^0$-lines are the world lines of the matter.

In order to investigate the motion of the matter, we have to integrate Eqs. (5.6). Repeating the same reasoning as in Sec. 5 and using the same notations we find that the equations

$$\dot{\eta}^1 = (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}} \frac{1+s}{(1-s^2)^{\frac{1}{2}}}\eta^2,$$
$$\dot{\eta}^2 = -(\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}} \frac{1-s}{(1-s^2)^{\frac{1}{2}}}\eta^1, \qquad (8.6)$$
$$\dot{\eta}^3 = 0$$

describe the motion of the matter with respect to the 3-dimensional vector frame Y_1, Y_2, Y_3. The orbits of the neighboring particles are given by

$$\left(\frac{\eta^1}{(1+s)^{\frac{1}{2}}}\right)^2 + \left(\frac{\eta^2}{(1-s)^{\frac{1}{2}}}\right)^2 = A^2, \quad \eta^3 = B, \quad (8.7)$$

which are ellipses in the Y_1, Y_2 plane. The main axes of the ellipse lie in the Y_1 and Y_2 directions.

The axes of the ellipse do not rotate, since from the equations

$$\dot{Y}_a \equiv \nabla_{Y_0}(Y_a) = \Gamma_{0a}{}^b\, Y_b$$

and from (8.2) it follows that

$$\dot{Y}_0 = 0, \quad \dot{Y}_1 = 0, \quad \dot{Y}_2 = 0, \quad \dot{Y}_3 = 0.$$

Therefore the frame Y_1, Y_2, Y_3 is parallel-propagated along the x^0 lines and can therefore be chosen as the inertial compass. This gives a characterization for the frame Y_0, Y_1, Y_2, Y_3 by the motion of the matter.

Another way to bring Y_1, Y_2, Y_3 in connection with the motion of the matter is to compute the tensor of shear and the vector of rotation. Along the lines explained in Sec. 5 and using the same notation, we find that

$$\sigma = (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}}s(1-s^2)^{-\frac{1}{2}}[2^{-\frac{1}{2}}(Y_1+Y_2)\otimes 2^{-\frac{1}{2}}(Y_1+Y_2) - 2^{-\frac{1}{2}}(Y_1-Y_2)\otimes 2^{-\frac{1}{2}}(Y_1-Y_2)], \quad (8.8)$$

that is, the eigenvalues of σ are

$$\pm\, (\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}}s(1-s^2)^{-\frac{1}{2}} \qquad (8.9)$$

and the corresponding eigenvectors are

$$2^{-\frac{1}{2}}(Y_1 \pm Y_2). \qquad (8.10)$$

The vector of rotation is given by

$$V = -(\tfrac{1}{2}\kappa\rho)^{\frac{1}{2}}s(1-s^2)^{-\frac{1}{2}}Y_3. \qquad (8.11)$$

The shear is vanishing for $s = 0$; therefore, we have the Gödel cosmos at this value of the parameter s as we already stated at the end of Sec. 7.

This concludes the characterization of the frame Y_0, Y_1, T_2, Y_3 by the motion of the matter.

Using (2.36), (2.37), (2.39) and the formulas

$$C_{abcd} = R_{abcd} - E_{abcd} - \tfrac{1}{12}Rg_{abcd},$$

where

$$E_{abcd} = \tfrac{1}{2}(S_{ad}g_{bc} - S_{ac}g_{bc} + g_{ad}S_{bc} - g_{ac}S_{bd}),$$
$$S_{ab} = R_{ab} - \tfrac{1}{4}Rg_{ab}, \qquad (8.12)$$

and

$$g_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd},$$

we can calculate the components of the Weyl tensor field. The nonvanishing components are

$$C_{2323} = -C_{1010} = \tfrac{1}{6}\kappa\rho, \quad C_{3131} = -C_{2020} = \tfrac{1}{6}\kappa\rho,$$
$$C_{1212} = -C_{3030} = -\tfrac{1}{3}\kappa\rho.$$

This is a type I e^2 Weyl tensor[10] (type D) and Y_0, Y_1, Y_2, Y_3 are the Weyl vectors. We can give for (7.6) a similar description as we gave to (4.28) toward the end of Sec. 6.

We can think of the space–time (7.6) as a 1-parametric family of 3-dimensional hypersurfaces—x^3 being the parameter. These hypersurfaces are generated by H^3, and all have the same geometry. The x^3 lines are perpendicular to these hypersurfaces, which are embedded in (7.6) such that Z_0, Z_1, Z_2 are parallelly propagated along the x^3 lines. The x^3 lines are spacelike.

From this, one sees that (7.6) is intrinsically similar than (4.28). One can use these models to study the motion of rotation within the general theory of relativity.

In a forthcoming paper, we discuss singularly the Class IV universes which then exhaust all the possibilities for homogeneous universes with dust within the framework of general relativity.

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